

Regularity of the steering control for systems with persistent memory ^{*}

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Abstract

The following fact is known for large classes of distributed control systems: when the target is regular, there exists a regular steering control. This fact is important to prove convergence estimates of numerical algorithms for the approximate computation of the steering control.

We extend this property to a class of systems with persistent memory of Maxwell-Boltzmann type.

1 Introduction

We study the following system where $x \in (0, \pi)$ and $t > 0$:

$$\begin{cases} w''(x, t) = w_{xx}(x, t) + \int_0^t M(t-s)w_{xx}(x, s) \, ds, & w(0, t) = f(t), \quad w(\pi, t) = 0 \\ w(x, 0) = 0, & w'(x, 0) = 0. \end{cases} \quad (1)$$

We assume $M(t) \in H^2(0, T)$ and $f(t) \in L^2(0, T)$ for every $T > 0$. As proved for example in [3], $w(x, t) \in C([0, T]; L^2(0, \pi)) \cap C^1([0, T]; H^{-1}(0, \pi))$ and for every $(\xi, \eta) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ and $T > 2\pi$, there exists $f \in L^2(0, T)$ such that $w(T) = \xi$, $w'(T) = \eta$. We prove:

Theorem 1. *Let $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$ and let $T > 2\pi$. There exists a steering control $f \in H_0^1(0, T)$.*

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The proof is in Section 2. Extensions to the case $x \in \Omega \subseteq \mathbb{R}^d$, $d > 1$, is reserved to future investigations.

We conclude this introduction with few comments. First we note that system (1) is often encountered in the study of viscoelasticity and diffusion equations with memory. When $M(t) = 0$ of course it reduces to the string equation. In the case of the wave equation (even when x in regions of \mathbb{R}^d , $d > 1$) theorem 1 is known. The proof that we give, based on moment methods, shows in particular controllability (in $H_0^1(0, \pi) \times L^2(0, \pi)$) of the cascade connection of system (1) with an integrator. We refer to [5, Ch. 11] and references therein for these facts.

2 The proof of Theorem 1

The following computations are a bit simplified if we integrate the first equation of (1) on $[0, t]$ and we write it in the equivalent form (here $N(t) = 1 + \int_0^t M(s) ds$)

$$w'(x, t) = \int_0^t N(t-s) w_{xx}(x, s) ds, \quad w(x, 0) = 0, \quad w(0, t) = f(t), \quad w(\pi, t) = 0. \quad (2)$$

We use the orthonormal basis of $L^2(0, \pi)$ whose elements are $\Phi_n = \sqrt{(2/\pi)} \sin nx$, $n \in \mathbb{N}$, and we expand

$$w(x, t) = \sum_{n \in \mathbb{N}} \Phi_n(x) w_n(t), \quad w_n(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi \Phi_n(x) w(x) dx.$$

Then $w_n(x, t)$ must satisfy

$$w'_n(t) = -n^2 \int_0^t N(t-s) w_n(s) ds + n \int_0^t N(t-s) \left(\sqrt{2/\pi} f(s) \right) ds.$$

The function $\sqrt{2/\pi} f$ will be renamed f .

Let $z_n(t)$ solve

$$z'_n(t) = -n^2 \int_0^t N(t-s) z_n(s) ds, \quad z_n(0) = 1. \quad (3)$$

We have (see [1])

$$\begin{aligned}
w_n(t) &= n \int_0^t \left(\int_0^{t-s} N(t-s-\tau) z_n(\tau) d\tau \right) f(s) ds = \\
&= \frac{1}{n} \int_0^t \left(\frac{d}{ds} z_n(t-s) \right) f(s) ds, \tag{4}
\end{aligned}$$

$$w'_n(t) = n \int_0^t \left(-\frac{d}{ds} \int_0^{t-s} N(t-s-\tau) z_n(\tau) d\tau \right) f(s) ds. \tag{5}$$

We require that a target $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$ is reached at time T , i.e. we require $(w(T), w'(T)) = (\xi, \eta)$.

The Fourier expansion of the targets is

$$\xi = \sum_{n=1}^{+\infty} \frac{\xi_n}{n} \Phi_n, \quad \text{and} \quad \eta = \sum_{n=1}^{+\infty} \eta_n \Phi_n, \quad (\{\xi_n\}, \{\eta_n\}) \in l^2(\mathbb{N}) \times l^2(\mathbb{N}).$$

So, controllability to (ξ, η) at time T is equivalent to the existence of a control $f \in L^2(0, T)$ such that $w_n(T) = \xi_n/n$, $w'_n(T) = \eta_n$ for every n . The expression we found for $w_n(t)$ and $w'_n(t)$ suggest that we investigate whether is it possible to solve this problem with

$$f(t) = \int_0^t g(s) ds, \quad g \in L^2(0, T). \tag{6}$$

If this is possible then we have the existence of an H^1 -steering control, and we get a steering control in $H_0^1(0, T)$ if we can find g which satisfies the additional condition

$$\int_0^T g(s) ds = 0. \tag{7}$$

We replace the expression (6) in $w_n(T)$ and $w'_n(T)$ and we integrate by parts. We see that f is an H^1 steering control to (ξ, η) if the following *moment problem* is solvable:

$$\xi_n = \int_0^T g(r) dr - \int_0^T z_n(T-s) g(s) ds, \tag{8}$$

$$\eta_n = \int_0^T \left[n \int_0^{T-s} N(T-s-r) z_n(r) dr \right] g(s) ds = \int_0^T g(T-s) \left(\frac{-z'_n(s)}{n} \right) ds. \tag{9}$$

We multiply equation (9) by i and we subtract it from (8). Furthermore we impose the additional condition (7). We find the moment problem:

$$\int_0^T Z_n(s) g(T-s) ds = c_0, \quad c_0 = \begin{cases} \xi_n - i\eta_n & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{10}$$

and $Z_n(t) = (z_n(s) + \frac{i}{n} z'_n(s))$ if $n > 0$, $Z_0(t) = 1$. In order to prove statement ?? of Theorem 1, we prove solvability of the moment problem (10).

We note that $\{c_n\}_{n>0}$ is an arbitray *complex valued* $l^2(\mathbb{N})$ sequence while g is real (when ξ and η are real). We reformulate the moment problem (10) with $n \in \mathbb{Z}$. We proceed as follows: for $n < 0$ we define:

$$z_n(t) = z_{-n}(t), \quad \Phi_n(x) = \Phi_{-n}(x), \quad Z_{-n}(t) = \bar{Z}_n(t).$$

Proceeding as in [3, Lemma 5.1] we can see that the moment problem (10) can be equivalently studied with $n \in \mathbb{Z}$ and g complex valued.

Our goal is the proof that the moment problem (10), $n \in \mathbb{Z}$, is solvable. Even more, we prove that $\{Z_n(t)\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$, provided that $T > 2\pi$. This imply the additional information that *the solution* $g \in L^2(0, T)$ *of minimal norm depends continuously on the target* $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$. Integrating this function g as in (6) we get the steering control f of minimal norm in $H_0^1(0, T)$ and so *the solution* $f \in H_0^1(0, T)$ *of minimal norm depends continuously on the target* $(\xi, \eta) \in H_0^1(0, \pi) \times L^2(0, \pi)$.

2.1 The proof that $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$, $T > 2\pi$

The proof that $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$, $T > 2\pi$, is divided in two steps: in the first one we show that the sequence $\{Z_n\}_{n \in \mathbb{Z} \setminus \{0\}}$ is a Riesz sequence in $L^2(0, T)$. Then we will prove that $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $L^2(0, T)$ too. In the proof we use the following definitions and results (see [3, Chp. 3]): a sequence $\{x_n\}$ in a Hilbert space H is:

- a *Riesz sequence* when it is the image of an orthonormal sequence under a linear bounded and boundedly invertible transformation;
- ω -*independent* when the following holds: if $\{\alpha_n\} \in l^2$ and if $\sum_{n=1}^{+\infty} \alpha_n x_n = 0$ (convergence in the norm of H) then $\{\alpha_n\} = 0$.

Let $\{x_n\}$ be a Riesz sequence in the Hilbert space H and let $\{y_n\}$ be quadratically close to $\{x_n\}$, i.e. $\sum_{n=1}^{+\infty} \|x_n - y_n\|_H^2 < +\infty$. Then there exists N such that $\{y_n\}_{|n|>N}$ is a Riesz sequence. If furthermore $\{y_n\}$ is ω -independent then it is a Riesz sequence too.

We introduce the notation and $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$.

Step 1: $\{Z_n\}_{n \in \mathbb{Z}'}$ is a Riesz sequence in $L^2(0, T)$, $T > 2\pi$ This part of the proof is contained in [2]. For completeness, we report the proof in the form we need here.

We put $N'(0) = \gamma$. Using [2, Lemmas 5.2 and 5.5] we get that for every $T > 0$ there exists C such that

$$\sum_{n \in \mathbb{Z}'} \|Z_n(t) - e^{\gamma t} e^{int}\|_{L^2(0,T)}^2 \leq C. \quad (11)$$

Then there exists $N > 0$ such that $\{Z_n\}_{|n| \geq N}$ is a Riesz sequence in $L^2(0, T)$.

We prove that $\{Z_n\}_{n \in \mathbb{Z}'}$ is ω -independent i.e. we prove that $\{\alpha_n\}_{n \in \mathbb{Z}'} = 0$ when $\{\alpha_n\} \in l^2(\mathbb{Z}')$ and

$$\sum_{n \in \mathbb{Z}'} \alpha_n Z_n = 0 \quad \text{i.e.} \quad \sum_{n \in \mathbb{Z}'} \alpha_n \left(z_n + \frac{i}{n} z'_n \right) = 0 \quad (12)$$

(this proof can be found in [2] althout not in such direct form, and it is reported for completeness).

Using $T > 2\pi$ and [3, Lemma 3.4] applied twice it is possible to prove that $\alpha_n = \frac{\gamma_n}{n^2}$ with $\{\gamma_n\} \in l^2(\mathbb{Z}')$ (see also [1]). This fact justifies the termwise differentiation of the series (12). Using

$$z_n''(t) = -n^2 N(t) - n^2 \int_0^t N(t-s) z'_n(s) ds \quad (13)$$

we get

$$\int_0^t N(t-s) \left[\sum_{n \in \mathbb{Z}'} \gamma_n \left(z_n(s) + \frac{i}{n} z'_n(s) \right) \right] ds - iN(t) \left[\sum_{n \in \mathbb{Z}'} \frac{\gamma_n}{n} \right] = 0. \quad (14)$$

Computing with $t = 0$ we see that $\sum_{n \in \mathbb{Z}'} n \alpha_n = \sum_{n \in \mathbb{Z}'} \frac{\gamma_n}{n} = 0$ and so, using $N(0) \neq 0$, we get

$$\sum_{n \in \mathbb{Z}'} [n^2 \alpha_n z_n(s) + i n \alpha_n z'_n(s)] = 0 \quad \text{hence} \quad \sum_{n \neq \pm 1, n \in \mathbb{Z}'} \alpha_n (n^2 - 1) \left[z_n + \frac{i z'_n}{n} \right] = 0.$$

Note that $\{\alpha_n (n^2 - 1)\} = \{\alpha_n^{(1)}\} \in l^2(\mathbb{Z}')$. Hence we can start a bootstrap argument and repeat this procedure. After at most $2N$ iteration of the process we get

$$\sum_{|n| > N} \alpha_n^{(N)} Z_n = 0$$

and so $\alpha_n^{(N)} = 0$ when $|n| > N$ since we noted that $\{Z_n\}_{|n| > N}$ is a Riesz sequence in $L^2(0, T)$. We have $\alpha_n^{(N)} = 0$ if and only if $\alpha_n = 0$ and this shows that the series (12) is a finite sum, $\sum_{n \in \mathbb{Z}', |n| \leq N} \alpha_n Z_n = 0$. The proof is now finished since it is easy to prove, as in [3, 4], that *the sequence $\{Z_n(t)\}_{n \in \mathbb{Z}'}$ is linearly independent.*

Step 2: $\{Z_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence Of course, $\{Z_n\}_{n \in \mathbb{Z}}$ is quadratically close to $\{e^{\gamma t} e^{int}\}_{n \in \mathbb{Z}}$. It remains to prove ω -independence, when $T > 2\pi$. We prove $\{\alpha_n\}_{n \in \mathbb{Z}} = 0$ when $\{\alpha_n\} \in l^2(\mathbb{Z})$ and

$$\alpha_0 + \sum_{n \in \mathbb{Z}'} \alpha_n Z_n = 0. \quad (15)$$

Using that constant functions belong to H^1 and [3, Lemma 3.4] applied twice we see that $\alpha_n = \gamma_n/n^2$, $\{\gamma_n\} \in l^2$. So, we can compute termwise the derivatives of both the sides of (15) and we get

$$\sum_{n \in \mathbb{Z}'} \alpha_n \left(z'_n(t) + \frac{i}{n} \left[-n^2 N(t) - n^2 \int_0^t N(t-s) z'_n(s) \, ds \right] \right) = 0. \quad (16)$$

Computing with $t = 0$ we get $\sum_{n \in \mathbb{Z}'} \alpha_n n = 0$. Then (using (3)) the equation (16) becomes

$$\int_0^t N(t-s) \left[\sum_{n \in \mathbb{Z}'} (\alpha_n n^2 z_n(s) + i \alpha_n n z'_n(s)) \right] \, ds = 0$$

so that (using again $N(0) \neq 0$ and $\{\alpha_n n^2\} \in l^2$)

$$\sum_{n \in \mathbb{Z}'} \alpha_n n^2 \left[z_n(t) + i \frac{1}{n} z'_n(t) \right] = \sum_{n \in \mathbb{Z}'} \alpha_n n^2 Z_n(t) = 0. \quad (17)$$

The fact that $\{Z_n(t)\}_{n \in \mathbb{Z}'}$ is a Riesz sequence implies that $\{\alpha_n\} = 0$ and so also $\alpha_0 = 0$, as we wanted to prove.

This ends the proof of Theorem 1.

Remark: The proof we have given shows that the cascade system

$$\begin{cases} w''(x, t) = w_{xx}(x, t) + \int_0^t M(t-s) w_{xx}(x, s) \, ds, & w(0, t) = y(t), \quad w(\pi, t) = 0 \\ y'(t) = g(t) w(x, 0) = 0, & w'(x, 0) = 0. \end{cases}$$

is controllable in $H_0^1(0, \pi) \times L^2(0, \pi)$, using the controls $g \in L_0(0, T)$ where

$$L_0(0, T) = \left\{ g \in L^2(0, T), \quad \int_0^T g(s) \, ds = 0 \right\}.$$

We refer to [5, Ch. 11] for the analysis of controllability of interconnected systems.

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